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Exact Complex Invariants in 3-D for Classical System

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ABSTRACT

Attempts are made to construct three-dimensional complex dynamical invariant by using Lie- algebraic method to study complex dynamical systems on the extended complex phase. Such invariants play an important role in the analysis of complex trajectories with regard to the calculation of semi-classical coherent state propagator in the path integral method.

Keywords: Complex Hamiltonian, Exact complex invariant.

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1. INTRODUCTION

Complex invariants are interesting field of research after real invariant. Since we are dealing with complex dynamical systems more or often in variety of field in physics [1, 2]. Several methods of complexification of systems are used in past. One type of complexification of the phase space in the form $z = p + i\omega x, \bar{z} = p - i\omega x$ has been widely discussed in the context of quantum mechanics and field theory. A possible generalized version obtained by introducing two complex variables $u = x/b + ip/c$ and $v = x/b - ip/c$, where b and c could be complex numbers. One can note that in both the above cases complexity arises through parameter space. In another approach, called ``complex scaling" method, one replaces $x \to xe^{i\theta}$ and writes the complex-scaled Hamiltonian operator for a given real H as

$$H_\theta = S^{-1}(\theta)\hat{H}S(\theta)$$

where S is the complex scaling operator defined by $S = exp[i\theta x(d/dx)]$ such that $Sf(x) = f(xe^{i\theta})$ for an analytical function $f(x)$. However methods used by Xavier and Aguiar [3] to develop an algorithm for the computation of the semi-classical coherent-state propagator has been used by many researchers, where transformations of phase space are as

$$x = x_1 + ip_2; \quad p = p_1 + ix_2.$$

Very recently Kaushal and his co-workers [4, 5] has investigated the construction of complex invariants of one-dimensional complex Hamiltonian systems on the extended complex phase plane (ECPS), characterized by $x = x_1 + ip_2$ and $p = p_1 + ix_2$. In this approach both x and p are separately made complex by extending each of them to the corresponding complex planes, i.e. inserting an imaginary component in each. From physics point of view p_2 and x_2 can be regarded as fictitious/spurious components of momentum and coordinate respectively and their presence in the above transformation equations as such allow the introduction of some sort of coordinate-momentum coupling of the dynamical system. this also has been extended to two-dimensions by Misra *et* *al* [6, 7]. Here, in the present work we carry out the extended phase plane approach to obtain exact complex invariants of a three-dimensional classical dynamical system .

2. THE METHOD

Consider a 3D real phase space (x, y, z, p_x, p_y, p_z) , which may be transformed into a complex space $(x_1, p_3, x_2, p_4, p_1, x_3, p_2, x_4)$, by defining position and momenta variables as

$$x = x_1 + ip_4; \quad y = x_2 + ip_5; \quad z = x_3 + ip_5;$$

$$p_x = p_1 + ix_4; \quad p_y = p_2 + ix_5; \quad p_z = p_3 + ix_6; \qquad (1)$$

Of course, the presence of variables $(x_4, x_5, x_6, p_5, p_5, p_6)$ in the above transformation eq.(1), can be regarded as some sort of coordinate-momentum interaction of the dynamical system.

The Hamiltonian $H(x, y, z, p_x, p_y, p_z)$ of a 3D system in complex space can be expressed, using eq.(1), as $H = H_1(x_1, p_4, x_2, p_5, x_3, p_6, p_1, x_4, p_2, x_5, p_3, x_6) + iH_2(x_1, p_4, x_2, p_5, x_3, p_6, p_1, x_4, p_2, x_5, p_3, x_6)$ From eq.(1) one can easily obtain

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x_1} - i\frac{\partial}{\partial p_4}; \quad \frac{\partial}{\partial p_x} = \frac{\partial}{\partial p_1} - i\frac{\partial}{\partial x_3}; \frac{\partial}{\partial y} = \frac{\partial}{\partial x_2} - i\frac{\partial}{\partial p_4};$$

$$\frac{\partial}{\partial p_y} = \frac{\partial}{\partial p_2} - i\frac{\partial}{\partial x_4}; \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial x_3} - i\frac{\partial}{\partial p_6}; \quad \frac{\partial}{\partial p_z} = \frac{\partial}{\partial p_3} - i\frac{\partial}{\partial x_6}; \qquad (2)$$

The Hamilton's equations of motion for complex H , eq.(2), can be written as

$$\dot{x}_1 = \frac{\partial H_1}{\partial p_1} + \frac{\partial H_2}{\partial x_4}; \quad \dot{p}_4 = \frac{\partial H_2}{\partial p_1} - \frac{\partial H_1}{\partial x_4}; \quad \dot{x}_2 = \frac{\partial H_1}{\partial p_2} + \frac{\partial H_2}{\partial x_5};$$

$$\dot{p}_5 = \frac{\partial H_2}{\partial p_2} - \frac{\partial H_1}{\partial x_5}; \quad \dot{x}_3 = \frac{\partial H_1}{\partial p_3} + \frac{\partial H_2}{\partial x_6}; \quad \dot{p}_6 = \frac{\partial H_2}{\partial p_2} - \frac{\partial H_1}{\partial x_6};. \qquad (3)$$

If the H , eq.(2), is an analytic function of complex variables, then H_1 and H_2 satisfy the Cauchy-Riemann conditions and after invoking such analyticity conditions, eq.(3) reduces

$$\dot{x}_1 = 2\frac{\partial H_1}{\partial p_1}; \dot{p}_1 = -2\frac{\partial H_1}{\partial x_1}; \dot{x}_2 = 2\frac{\partial H_1}{\partial p_2}; \dot{p}_2 = -2\frac{\partial H_1}{\partial x_2}; \dot{x}_3 = 2\frac{\partial H_1}{\partial p_3}; \dot{p}_3 = -2\frac{\partial H_1}{\partial x_3};$$

$$\dot{x}_4 = 2\frac{\partial H_1}{\partial p_4}; \dot{p}_4 = -2\frac{\partial H_1}{\partial x_4}; \dot{x}_5 = 2\frac{\partial H_1}{\partial p_5}; \dot{p}_5 = -2\frac{\partial H_1}{\partial x_5}; \dot{x}_6 = 2\frac{\partial H_1}{\partial p_6}; \dot{p}_6 = -2\frac{\partial H_1}{\partial x_6}; \qquad (4)$$

Note that $(x_1, p_1), (x_2, p_2), (x_3, p_3), (x_4, p_4), (x_5, p_5),$ and (x_6, p_6) constitute canonical pairs.

Now consider a complex phase space function $I(x, y, z, p_x, p_y, p_z, t)$ as $I = I_1 + iI_2$. Thus for function I to be the TD dynamical invariant of the system in complex phase space, then this must conform the following invariance condition:

$$\frac{dI}{dt} = \frac{\partial I}{\partial t} + [I, H] = 0, \tag{5}$$

where $[.,.]$ is the Poisson bracket, which in view of the definition, eq.(1), turns out to be

$$[A,B]_{(x,p)} = [A,B]_{(x_1,p_1)} - i[A,B]_{(x_1,x_4)} - i[A,B]_{(p_4,p_1)} - [A,B]_{(p_4,x_4)} + [A,B]_{(x_2,p_2)} - i[A,B]_{(x_2,x_5)}$$
$$- i[A,B]_{(p_5,p_2)} - [A,B]_{(p_5,x_5)} + [A,B]_{(x_3,p_3)} - i[A,B]_{(x_3,x_6)} - i[A,B]_{(p_6,p_3)} - [A,B]_{(p_6,x_6)} \tag{6}$$

which indicates that the computation of Poisson bracket in case of complex Hamiltonian systems becomes a bit tedious. With a view to demonstrate the underlying elegance of the Lie-algebraic approach, we briefly describe the method to construct complex invariants of classical dynamical systems.

In the Lie-algebraic approach, [8] one can express the complex Hamiltonian $H(x, y, z, p_x, p_y, p_z)$ of the system as

$$H = \sum_n h_n(t)\Gamma_n(x, y, z, p_x, p_y, p_z), \tag{7}$$

where the set of functions $\{\Gamma_1, ..., \Gamma_n\}$ are not explicitly TD and $h_n(t)$'s are complex coefficient functions of time. The Γ_n's in eq.(7) generate a closed dynamical algebra, implies $[\Gamma_n, \Gamma_m] = \sum_l C^l_{nm}\Gamma_l$, where C^l_{nm} are the complex structure constants of the algebra. If the Γ_n's are not sufficient to close the algebra then, the set of Γ_n's must be extended by adding new Γ_l's, such that $\Gamma_l = [\Gamma_n, \Gamma_m]$, until the closure is obtained. The additional $h_l(t)$'s are taken to be zero. Since the complex dynamical invariant I is also a part of Lie algebra, then one can write this as

$$I(t) = \sum_k \lambda_k(t)\Gamma_k(x, y, z, p_x, p_y, p_z), \tag{8}$$

where $\lambda_k(t)$'s are TD (time dependent) complex coefficients. Thus by using eq.(7) and eq.(8) for H and I respectively in eq.(5), we get a system of linear, first order differential equations, namely

$$\dot{\lambda}_r + \sum_n [\sum_m C^r_{nm} h_m(t)]\lambda_n = 0, \tag{9}$$

in λ_n's. Thus, the solutions of these differential equations in turn provide classical complex invariants of a given system. In the next subsection we shall use the prescription given above to obtain complex invariant of a classical complex Hamiltonian system.

3. EXAMPLE

Consider a simple harmonic oscillator in 3-dimension, whose Hamiltonian is given by

$$H = \frac{1}{2}[p_x^2 + p_y^2 + p_y^2] + \frac{w^2}{2}[x^2 + y^2 + z^2]. \tag{10}$$

Using eq.(1), the above Hamiltonian can be expressed as

$$H=\frac{p_1^2}{2}-\frac{x_4^2}{2}+\frac{p_2^2}{2}-\frac{x_5^2}{2}+\frac{p_3^2}{2}-\frac{x_6^2}{2}+i[p_1x_4+p_2x_5+p_3x_6+x_1p_4+x_2p_5+x_3p_6]$$

$$+\frac{1}{2}x_1^2-\frac{p_4^2}{2}+\frac{x_2^2}{2}-\frac{p_5^2}{2}+\frac{x_3^2}{2}-\frac{p_6^2}{2}=\sum_{m=1}^{18}h_m(t)\Gamma_m(x,p) \quad (11)$$

and the various Γ's and h(t)'s for the above complex H are given as

$$\Gamma_1=\frac{1}{2}p_1^2;\Gamma_2=\frac{1}{2}x_4^2;\Gamma_3=\frac{1}{2}p_2;\Gamma_4=\frac{1}{2}x_5^2;\Gamma_5=\frac{1}{2}p_3^2;\Gamma_6=\frac{1}{2}x_6^2;\Gamma_7=p_1x_4;\Gamma_8=p_2x_5;\Gamma_9=x_6p_3;$$

$$\Gamma_{10}=x_1p_4;\Gamma_{11}=x_2p_5;\Gamma_{12}=x_3p_4;\Gamma_{13}=\frac{1}{2}x_1^2;\Gamma_{14}=\frac{1}{2}p_4;\Gamma_{15}=\frac{1}{2}x_2;\Gamma_{16}=\frac{1}{2}p_5;\Gamma_{17}=\frac{1}{2}x_3^2;\Gamma_{18}=\frac{1}{2}p_6^2;\ with$$

$$h_{1,3,5}=1 \quad h_{2,4,6}=-1,\quad h_{7,8,9}=i,\quad h_{10,11,12}=iw^2,\quad h_{13,15,17}=w^2,\quad h_{14,16,18}=-w^2. \quad (12)$$

The dynamical algebra in this case is not closed. To find closure property for the above system, we add twelve more phase space functions (Γ_l)'s. The additional (Γ_l)'s are as follow

$$\Gamma_{19}=p_6x_6;\quad \Gamma_{20}=p_3x_3;\quad \Gamma_{21}=p_3p_6;\quad \Gamma_{22}=x_3x_6;\ \Gamma_{23}=x_5p_5;\ \Gamma_{24}=x_2x_5;$$

$$\Gamma_{25}=x_1x_4;\ \Gamma_{26}=p_2x_2;\ \Gamma_{27}=p_2p_5;\ \Gamma_{28}=x_4p_4;\Gamma_{29}=p_1p_4;\quad \Gamma_{30}=p_1x_1; \quad (13)$$

with corresponding $h_l(t)=0$. Now in the light of Poisson bracket for complex systems, eq.(6), we get large number of nonvanishing Poisson brackets, namely

$$[\Gamma_1,\Gamma_{10}]=i\Gamma_{30}-\Gamma_{29};\ [\Gamma_1,\Gamma_{13}]=-\Gamma_{30};[\Gamma_1,\Gamma_{14}]=i\Gamma_{29};\ [\Gamma_1,\Gamma_{25}]=-\Gamma_7;[\Gamma_1,\Gamma_{28}]=i\Gamma_7;$$

$$[\Gamma_1,\Gamma_{29}]=2i\Gamma_1;[\Gamma_1,\Gamma_{30}]=-2i\Gamma_1;[\Gamma_2,\Gamma_{13}]=i\Gamma_{25};\ [\Gamma_2,\Gamma_{10}]=\Gamma_{25}+i\Gamma_{28};\ [\Gamma_2,\Gamma_{14}]=\Gamma_{28};$$

$$[\Gamma_2,\Gamma_{25}]=2i\Gamma_2;[\Gamma_2,\Gamma_{28}]=2\Gamma_2;\ [\Gamma_2,\Gamma_{29}]=\Gamma_7;\ [\Gamma_2,\Gamma_{30}]=i\Gamma_7;[\Gamma_3,\Gamma_{11}]=-\Gamma_{27}+i\Gamma_{26};$$

$$[\Gamma_3,\Gamma_{15}]=-\Gamma_{26};\ [\Gamma_3,\Gamma_{16}]=i\Gamma_{27};\ [\Gamma_3,\Gamma_{23}]=i\Gamma_8;[\Gamma_3,\Gamma_{24}]=\Gamma_8;\ [\Gamma_3,\Gamma_{26}]=-2\Gamma_3;[\Gamma_3,\Gamma_{27}]=2i\Gamma_3;$$

$$[\Gamma_4,\Gamma_{11}]=i\Gamma_{23}+\Gamma_{24};\ [\Gamma_4,\Gamma_{15}]=\Gamma_{24}+i\Gamma_{23};\ [\Gamma_4,\Gamma_{16}]=\Gamma_{23};[\Gamma_4,\Gamma_{23}]=2\Gamma_4;\ [\Gamma_4,\Gamma_{24}]=2i\Gamma_4;$$

$$[\Gamma_4,\Gamma_{26}]=i\Gamma_8;\ [\Gamma_4,\Gamma_{27}]=-\Gamma_8;[\Gamma_5,\Gamma_{12}]=-\Gamma_{21}+i\Gamma_{20};\ [\Gamma_5,\Gamma_{17}]=-\Gamma_{20};\ [\Gamma_5,\Gamma_{18}]=i\Gamma_{21};$$

$$[\Gamma_5,\Gamma_{19}]=+i\Gamma_9;\ [\Gamma_5,\Gamma_{20}]=-2\Gamma_5;[\Gamma_5,\Gamma_{21}]=2i\Gamma_5;\ [\Gamma_5,\Gamma_{22}]=-\Gamma_9;[\Gamma_6,\Gamma_{12}]=i\Gamma_{19}+\Gamma_{22};$$

$$[\Gamma_6,\Gamma_{17}]=i\Gamma_{18}-\Gamma_{20};[\Gamma_6,\Gamma_{18}]=\Gamma_{19};[\Gamma_6,\Gamma_{19}]=2\Gamma_6;\ [\Gamma_6,\Gamma_{20}]=i\Gamma_9;[\Gamma_6,\Gamma_{21}]=\Gamma_9;\ [\Gamma_6,\Gamma_{22}]=2\Gamma_6;$$

$$[\Gamma_7,\Gamma_{10}]=i\Gamma_{29}+\Gamma_{30}-\Gamma_{28}+i\Gamma_{25};\ [\Gamma_7,\Gamma_{13}]=\Gamma_{30}-i\Gamma_{25};[\Gamma_7,\Gamma_{14}]=i\Gamma_{28}+\Gamma_{29};$$

$$[\Gamma_7,\Gamma_{25}]=i\Gamma_7-2\Gamma_2;\ [\Gamma_7,\Gamma_{28}]=2i\Gamma_2+\Gamma_7;\ [\Gamma_7,\Gamma_{29}]=2\Gamma_1+i\Gamma_7;[\Gamma_7,\Gamma_{30}]=-\Gamma_7+2i\Gamma_1;$$

$$[\Gamma_8,\Gamma_{11}]=i\Gamma_{24}+\Gamma_{26}-\Gamma_{23}+i\Gamma_{27};\ [\Gamma_8,\Gamma_{15}]=-\Gamma_{24}+i\Gamma_{26};[\Gamma_8,\Gamma_{16}]=\Gamma_{23}+\Gamma_{27},$$

$[\Gamma_8,\Gamma_{23}]=\Gamma_8+2i\Gamma_4;\ [\Gamma_8,\Gamma_{24}]=i\Gamma_8-2\Gamma_4;[\Gamma_8,\Gamma_{26}]=-\Gamma_8+2i\Gamma_3;\ [\Gamma_8,\Gamma_{27}]=i\Gamma_8+2\Gamma_3;$

$[\Gamma_9,\Gamma_{12}]=i\Gamma_{21}+\Gamma_{20}-\Gamma_{19}+i\Gamma_{22};\ [\Gamma_9,\Gamma_{17}]=-\Gamma_{22}+i\Gamma_{20};[\Gamma_9,\Gamma_{18}]=i\Gamma_{19}+\Gamma_{21},$

$[\Gamma_9,\Gamma_{19}]=2i\Gamma_6+\Gamma_9;\ [\Gamma_9,\Gamma_{20}]=-\Gamma_9+2i\Gamma_5;\ [\Gamma_9,\Gamma_{21}]=2\Gamma_5+i\Gamma_9;[\Gamma_9,\Gamma_{22}]=i\Gamma_9-2\Gamma_6;$

$[\Gamma_{10},\Gamma_{25}]=-i\Gamma_{10}-2\Gamma_3;\ [\Gamma_{10};\Gamma_{28}]=+2i\Gamma_{14}-\Gamma_{10};[\Gamma_{10},\Gamma_{29}]=2\Gamma_{14}-i\Gamma_{10};$
$[\Gamma_{10},\Gamma_{30}]=-2i\Gamma_{13}+\Gamma_{10};[\Gamma_{11},\Gamma_{23}]=-2i\Gamma_{16}-\Gamma_{11};\ [\Gamma_{11};\Gamma_{24}]=-i\Gamma_{18}-2\Gamma_{15};$
$[\Gamma_{11},\Gamma_{26}]=-2i\Gamma_{15}+\Gamma_{11};\ [\Gamma_{11},\Gamma_{27}]=2\Gamma_{16}-i\Gamma_{11};\ [\Gamma_{12},\Gamma_{19}]=-2i\Gamma_{18}-\Gamma_{12};$

$[\Gamma_{12},\Gamma_{20}]=\Gamma_{12}-2i\Gamma_{17};[\Gamma_{12},\Gamma_{21}]=2\Gamma_{18}-i\Gamma_{12},\ [\Gamma_{12},\Gamma_{22}]=-i\Gamma_{12}-2\Gamma_{17};$

$[\Gamma_{13},\Gamma_{25}]=-2i\Gamma_{13};\ [\Gamma_{13},\Gamma_{28}]=-i\Gamma_7;[\Gamma_{13},\Gamma_{29}]=\Gamma_{10};\ [\Gamma_{13},\Gamma_{30}]=2\Gamma_{13};$

$[\Gamma_{15},\Gamma_{23}]=2i\Gamma_8;\ [\Gamma_{15},\Gamma_{24}]=-2i\Gamma_{15};\ [\Gamma_{15},\Gamma_{26}]=2i\Gamma_{15};[\Gamma_{15},\Gamma_{27}]=\Gamma_{18};\ [\Gamma_{16},\Gamma_{23}]=2\Gamma_{16};$

$[\Gamma_{16},\Gamma_{26}]=2\Gamma_{15};\ [\Gamma_{16},\Gamma_{27}]=-2i\Gamma_{16};[\Gamma_{17},\Gamma_{19}]=-i\Gamma_{12};[\Gamma_{17},\Gamma_{20}]=2\Gamma_{17};\ [\Gamma_{17},\Gamma_{21}]=\Gamma_{12};$

$$[\Gamma_{17},\Gamma_{22}]=-2i\Gamma_{17};[\Gamma_{18},\Gamma_{19}]=-2\Gamma_{18};[\Gamma_{18},\Gamma_{20}]=-i\Gamma_{12}\ \ [\Gamma_{18},\Gamma_{21}]=-2i\Gamma_{18};[\Gamma_{18},\Gamma_{22}]=-\Gamma_{12} \tag{14}$$

Therefore, their use in eq.(9) yields the following set of Partial differential equations in λ's are :

$$\dot{\lambda}_1=4(i\lambda_{29}-\lambda_{30}),\qquad \dot{\lambda}_2=-4(i\lambda_{25}+\lambda_{28}), \tag{15}$$

$$\dot{\lambda}_3=4(i\lambda_{27}-\lambda_{26}),\qquad \dot{\lambda}_4=4(i\lambda_{24}-\lambda_{23}), \tag{16}$$

$$\dot{\lambda}_5=4(i\lambda_{21}-\lambda_{22}),\qquad \dot{\lambda}_6=-4(i\lambda_{22}+\lambda_{19}), \tag{17}$$

$$\dot{\lambda}_7=2(-\lambda_{25}+i\lambda_{28}-\lambda_{29}-i\lambda_{30}),\qquad \dot{\lambda}_8=2(i\lambda_{23}+\lambda_{24}-i\lambda_{26}-\lambda_{27}), \tag{18}$$

$$\dot{\lambda}_9=2(i\lambda_{19}-i\lambda_{20}-\lambda_{22}-\lambda_{21}),\qquad \dot{\lambda}_{10}=2\omega^2(\lambda_{25}-i\lambda_{28}+\lambda_{29}+i\lambda_{30}), \tag{19}$$

$$\dot{\lambda}_{11}=2\omega^2(i\lambda_{23}-\lambda_{24}+-i\lambda_{26}+\lambda_{16}),\qquad \dot{\lambda}_{12}=2\omega^2(-i\lambda_{19}+i\lambda_{20}+\lambda_{21}+\lambda_{22}), \tag{20}$$

$$\dot{\lambda}_{13}=-4\omega^2(i\lambda_{25}-\lambda_{30}),\qquad \dot{\lambda}_{14}=4\omega^2(i\lambda_{29}+\lambda_{28}), \tag{21}$$

$$\dot{\lambda}_{15}=-4\omega^2(\lambda_{24}-\lambda_{26}),\qquad \dot{\lambda}_{16}=-4\omega^2(\lambda_{23}-i\lambda_{27}), \tag{22}$$

$$\dot{\lambda}_{17}=4\omega^2(\lambda_{20}-i\lambda_{22}),\qquad \dot{\lambda}_{18}=4\omega^2(\lambda_{19}+i\lambda_{21}), \tag{23}$$

$$\dot{\lambda}_{19}=2\omega^2(\lambda_6+i\lambda_9-i\lambda_{12}+\lambda_{18}),\qquad \dot{\lambda}_{20}=2\omega^2(\lambda_5-i\lambda_9+i\lambda_{12}-\lambda_{17}), \tag{24}$$

$$\dot{\lambda}_{21} = 2\omega^2(i\lambda_5 + \lambda_9 - \lambda_{12} + i\lambda_{18}), \quad \dot{\lambda}_{22} = 2\omega^2(-i\lambda_6 + \lambda_9 - \lambda_{12} - i\lambda_{18}), \tag{25}$$

$$\dot{\lambda}_{23} = 2\omega^2(\lambda_4 + i\lambda_8 - i\lambda_{11} - \lambda_{16}), \quad \dot{\lambda}_{24} = 2\omega^2(-i\lambda_4 + \lambda_8 - \lambda_{11} - i\lambda_{15}), \tag{26}$$

$$\dot{\lambda}_{25} = 2\omega^2(\lambda_7 - i\lambda_2 - \lambda_{11} - i\lambda_{15}), \quad \dot{\lambda}_{26} = 2\omega^2(\lambda_4 - i\lambda_8 + i\lambda_{11} - \lambda_{15}). \tag{27}$$

$$\dot{\lambda}_{27} = 2\omega^2(i\lambda_3 + \lambda_8 - \lambda_{11} + i\lambda_{16}), \quad \dot{\lambda}_{28} = 2\omega^2(i\lambda_7 + \lambda_2 - i\lambda_{10} - \lambda_{14}), \tag{28}$$

$$\dot{\lambda}_{29} = 2\omega^2(i\lambda_1 + \lambda_7 - \lambda_{10} + i\lambda_{14}), \quad \dot{\lambda}_{30} = 2\omega^2(\lambda_1 + i\lambda_7 + i\lambda_{10} - \lambda_{13}), \tag{29}$$

In fact, to solve these 30 coupled PDEs for complex λ's is very difficult. Thus, here we make certain choices for λ's which facilitate to find solutions of above equations.

From eqs.(15) and (18), we get $2i\dot{\lambda}_7 = \dot{\lambda}_2 - \dot{\lambda}_1$, and if we consider $\lambda_7 = c_7$ (a constant), further $\lambda_1 = \lambda_2 = \eta(t)$, which immediately gives

$$\lambda_1 = \eta(t) + c_1 \quad , \quad \lambda_2 = \eta(t) + c_2, \tag{30}$$

From eqs.(16) and (18), we get $2i\dot{\lambda}_8 = \dot{\lambda}_4 - \dot{\lambda}_3$, and if we consider $\lambda_8 = c_8$ (a constant), further $\lambda_4 = \lambda_3 = \theta(t)$, which immediately gives

$$\lambda_4 = \theta(t) + c_4 \quad , \quad \lambda_3 = \theta(t) + c_3, \tag{31}$$

Again from eqs.(17) and (19), we get $2i\dot{\lambda}_9 = \dot{\lambda}_6 - \dot{\lambda}_5$, and if we consider $\lambda_9 = c_9$ (a constant), further $\lambda_6 = \lambda_5 = \zeta(t)$, which immediately gives

$$\lambda_5 = \zeta(t) + c_5 \quad , \quad \lambda_6 = \zeta(t) + c_6, \tag{32}$$

From eqs.(19), and (21), we get $2i\dot{\lambda}_{10} = -\dot{\lambda}_{13} + \dot{\lambda}_{14}$, and if we consider $\lambda_{10} = c_{10}$ (a constant), further $\lambda_{13} = \lambda_{14} = \delta(t)$, which immediately gives

$$\lambda_{13} = \delta(t) + c_{13} \quad , \quad \lambda_{14} = \delta(t) + c_{14}, \tag{33}$$

Again from eqs.(20), and (22), we get $2i\dot{\lambda}_{11} = \dot{\lambda}_{16} + \dot{\lambda}_{15}$, and if we consider $\lambda_{11} = c_{11}$ (a constant), further $-\lambda_{16} = \lambda_{15} = \nu(t)$, which immediately gives

$$\lambda_{15} = \nu(t) + c_{15} \quad , \quad \lambda_{16} = -\nu(t) + c_{16}, \tag{34}$$

From eqs.(20), and (23), we get $2i\dot{\lambda}_{12} = \dot{\lambda}_{18} - \dot{\lambda}_{17}$, and if we consider $\lambda_{12} = c_{12}$ (a constant), further $\lambda_{17} = \lambda_{18} = \xi(t)$, which immediately gives

$$\lambda_{17} = \xi(t) + c_{17} \quad , \quad \lambda_{18} = \xi(t) + c_{18}, \tag{35}$$

Now for finding the solutions of λ_{19} and λ_{22}, subtract i times eq.(25) from eq.(24) and after using eq.(35), we get

$$\dot{\lambda}_{22} - i\dot{\lambda}_{19} = 2i(\lambda_{17} + \lambda_{18}),\ or\ i\dot{\lambda}_{22} + \dot{\lambda}_{19} = -2(2\xi + c_{17} + c_{18}). \tag{36}$$

On the other hand, time derivative of eq.(15) is written as

$$\ddot{\lambda}_6 = 4(-i\lambda_{17} + \lambda_{18}) = \ddot{\phi}. \tag{37}$$

Hence using eq.(36) and (37), one immediately get

$$\lambda_{19} = -\frac{1}{8}(\dot{\phi} - 8\sigma_1) + c_{19},\ \lambda_{22} = \frac{i}{8}(\dot{\phi} - 8\sigma_1) + c_{22}, \tag{38}$$

where $\sigma_1 = \int (2\xi(t) + c_{17} + c_{18})dt$. Similarly, from eq.[(25)-(29)] we obtain solutions for $(\lambda_{20}, \lambda_{21})$, $(\lambda_{23}, \lambda_{24})$,$(\lambda_{25}, \lambda_{26})$, $(\lambda_{27}, \lambda_{28})$ and $(\lambda_{29}, \lambda_{30})$ respectively as

$$\lambda_{20} = -\frac{i}{8}(\dot{\phi} - 8\sigma_1) + c_{20}, \quad \lambda_{21} = \frac{1}{8}(\dot{\phi} + 8\sigma_1) + c_{21}, \quad \lambda_{23} = \frac{i}{8}(\dot{\chi} - 8\sigma_2) + c_{23},$$

$$\lambda_{24} = \frac{1}{8}(\dot{\chi} + 8\sigma_2) + c_{24}, \quad \lambda_{25} = \frac{1}{8}(\dot{\psi} + 8\sigma_3) + c_{25}, \quad \lambda_{26} = -\frac{i}{8}(\dot{\chi} - 8\sigma_2) + c_{26}$$

$$\lambda_{27} = \frac{1}{8}(\dot{\chi} + 8\sigma_2) + c_{27},\ \lambda_{28} = \frac{i}{8}(\dot{\psi} - 8\sigma_3) + c_{28}\ \ \lambda_{29} = \frac{1}{8}(\dot{\xi} + 8\sigma_3) + c_{29},\ \lambda_{30} = \frac{1}{8}(\dot{\psi} + 8\sigma_3) + c_{30} \tag{39}$$

where $\sigma_2 = \int (2\nu(t) + c_{15} + c_{16})dt$ and $\sigma_3 = \int (2\delta(t) + c_{13} + c_{14})dt$.

We have solved eqs.[(15)-(29)] in terms of arbitrary functions $\phi, \chi, \psi, \delta, \nu$ and ξ and complex constants, c_i's, $(i = 1,....,30)$. If one put back these solutions for λ_i, $(i = 1,....,30)$ in eqs.[(15)-(29)], we obtain a number of constraint relations among c_i's, and $\phi, \chi, \psi, \delta, \nu$ and ξ, which limit the choices of these arbitrary complex quantities. If we set c_i's equal to zero, then these relations determining arbitrary functions $\phi, \chi, \psi, \delta, \nu$ and ξ , are written as

$$\ddot{\phi} + 16\omega^2\phi = 0,\ \ddot{\chi} + 16\omega^2\chi = 0,\ \ddot{\psi} + 16\omega^2\psi = 0,$$

$$\ddot{\delta} + 8\omega^4\delta = 0,\ \ddot{\nu} + 8\omega^4\nu = 0,\ \ddot{\xi} + 8\omega^4\xi = 0. \tag{40}$$

Therefore, after substituting the solutions of λ_i's in the eq.(8), the complex invariant for a three-dimensional complex oscillator becomes

$$I = \frac{1}{2}\eta(p_1^2 + x_4^2) + \frac{1}{2}\theta(p_2^2 + x_5^2) + \frac{1}{2}\zeta(p_3^2 + x_6^2) + \frac{1}{2}\eta(x_1^2 + p_3^2) + \frac{1}{2}\delta(x_1^2 + p_4^2) + \frac{1}{2}\nu(x_2^2 + p_5^2) + \frac{1}{2}\xi(x_3^2 + p_6^2)$$
$$+ \frac{i}{8}(\dot{\phi} - 8\sigma_1)(x_3x_6 - p_3x_6) - \frac{1}{8}(\dot{\phi} + 8\sigma_1)(x_3p_3 + x_6p_6) - \frac{i}{8}(\dot{\chi} - 8\sigma_2)(x_2x_5 - p_2p_5) - \frac{1}{8}(\dot{\chi} + 8\sigma_2)(p_2x_2 + p_5x_5)$$
$$+ \frac{i}{8}(\dot{\xi} - 8\sigma_3)(x_1x_4 - p_1p_4) - \frac{1}{8}(\dot{\xi} + 8\sigma_3)(p_4x_4 + p_1x_1) \tag{41}$$

which conforms to condition eq.(7) in view of the Poisson bracket eq.(8), In this work, a modest attempt has been made to obtain exact complex second constant of motion of a three-dimensional system on an extended complex phase space characterized by eq.(1). The transformations (1) (or eq.(1)) had been a part of many studies [4, 5, 6, 7] and can give PT -symmetric Hamiltonian under certain boundary conditions. Lie algebraic method is used for construction of complex invariants, which has been used extensively for construction of TD real invariants of both classical and quantum systems. Since the degrees of freedom become just triple after complexification of the a system which make the construction of complex invariants a bit tedious in three dimensions.

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