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On rgw-Homeomorphism in Topological Space

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ABSTRACT

In this paper, we first introduce new class of homeomorphism called regular generalized weakly homeomorphism (briefly rgw-homeomorphism) in topological space and after that we compare this new class of homeomorphism with recently introduced generalized form of homeomorphism. Moreover, we study algebraic properties of the family of rgw - homeomorphism in topological space.

Keywords: rgw-homeomorphism; topological space.

Received on:9/6/2016 Accepted on:13/6/2016 Published online on:14-6-2016

1. INTRODUCTION

The homeomorphism in topological space play a significant role in the study of topology. The role of the homeomorphism in topology is same as like group isomorphisms in group theory, linear isomorphisms in linear algebra, biholomorphic maps in function theory, and isometries in Riemannian geometry. First we provide basic requirement for better understanding of rgw-Homeomorphism in topological space. In this paper, we define new class of homeomorphism called rgw-Homeomorphism and study the properties of this map and also compare this homeomorphism with well-known homeomorphism in order to have a better understanding of rgw-Homeomorphism. In last, we investigate algebraic properties of family of rgw-Homeomorphisms like equivalence relation, group under binary operation etc.

2. PRELIMINARIES

First, we recall some definition and results which will help to understand the new class of homomorphism. Closed set as like open set is one of important tool to define the topology on any set and generalization of closed set always give better conditions for finding weak or strong topologies. In this direction, first step was taken by L. Nevine [6] in 1970, who defined generalized closed in topological space and studied their properties. In past years, many researchers introduced various form of generalized closed which is defined under different conditions. In this section, we are presenting those generalized closed set which are directly related to our core discussion.

A non-empty subset A of topological space X is said to be

(1) Generalized closed (briefly g-closed) [6] set if and only if $Cl(A) \subseteq G$ whenever $A \subseteq G$ and G is open.

(2) Semi closed (briefly s-closed) [5] set if $Int(Cl(A)) \subset A$ and complement of A is called semi open (briefly s-open) set.

(3) Regular closed (briefly r-closed) [12] set if $A = Cl(Int(A))$.

(4) Regular generalized closed (briefly rg-closed) [10] set if $Cl(A) \subseteq G$ whenever $A \subseteq G$ and G is regular open in X.

(5) Weakly closed (briefly w-closed) [11] if $Cl(A) \subseteq G$ whenever $A \subseteq G$ and G is semi-open in X .

(6) Regular semi-open (briefly rs-closed) set [2] if there is a regular open set G such that $G \subset A \subset Cl(G)$.

(7) Regular generalized weakly closed (briefly rgw-closed) set [8] if $Cl(Int(A)) \subseteq G$ whenever $A \subseteq G$ and G is regular semi-open in X .

Complements of above mentioned closed sets are said to be their respective open sets. Now, we are giving following generalized continuous functions in term of above closed sets.

Definition 2.1. A function $f : (X, \tau_1) \rightarrow (Y, \tau_2)$ is said to be

(1) g-continuous [1] if $f^{-1}(H)$ is g$-\tau_1$-closed set of topological space X for every τ_2-closed set H of Y .

(2) rg-continuous [10] if $f^{-1}(H)$ is rg$-\tau_1$-closed set of topological space X for every τ_2-closed set H of Y .

(3) w-continuous [4] if $f^{-1}(H)$ is w$-\tau_1$-closed set of topological space X for every τ_2-closed set H of Y .

(4) rgw-continuous [9] if $f^{-1}(H)$ is rgw$-\tau_1$-closed in topological space X for every τ_2-closed set H in Y .

Recently, introduced following generalized form of homeomorphism as follows.

Definition 2.2. A bijection function f :$(X, \tau_1) \rightarrow (Y, \tau_2)$ is called

(1) Generalized homeomorphism [7] (briefly, g-homeomorphism) if both f and f^{-1} are g-continuous.

(2) Weakly homeomorphism [4] (briefly w-homeomorphism) if both f and f^{-1} are w-continuous.

(3) Semi-generalized homeomorphism [3] (briefly, sg-homeomorphism) if f is sg-continuous and sg-open.

(4) Generalized semi-homeomorphism [3] (briefly, gs-homeomorphism) if f is gs-continuous and gs-open.

3. THE rgw-HOMEOMORPHISM IN TOPOLOGICAL SPACE

Now, in this section we are going to define new class of homeomorphism called regular generalized weakly homeomorphism on topological space and further we will discuss some its interesting properties.

Definition 3.1. A bijection map f : $(X, \tau_1) \rightarrow (Y, \tau_2)$ is said to be rgw-homeomorphism if f and f^{-1} both are rgw-continuous map.

The family of all homeomorphism and rgw-homeomorphism of a topological space

(X, τ_1) onto itself is denoted by $h(X, \tau_1)$ and $rgw-h(X, \tau_1)$ respectively.

Definition 3.2. A bijection f : $(X, \tau_1) \rightarrow (Y, \tau_2)$ is said to be rgw∗-homeomorphism if both f and f^{-1} are rgw-irresolute.

The family of all rgw*-homeomorphism of a topological space (X, τ_1) onto itself is denoted by $rgw^*-h(X, \tau_1)$ respectively.

Now we compare new defined rgw-homeomorphism with recently introduced well-known generalized form of homeomorphism.

Theorem 3.3. Every homeomorphism is a rgw-homeomorphism.

Proof. Let f : $(X, \tau_1) \rightarrow (Y, \tau_2)$ is homeomorphism, then f and f^{-1} are continuous and also bijections. As every continuous map is rgw-continuous so f and f^{-1} are rgw-continuous. Therefore, f is rgw-homeomorphism. But the converse of this result is not true. Let $X = Y = \{p, q, r\}$, $\tau = \{\varphi, X, \{p\}\}$ and $\rho = \{\varphi, Y, \{q\}\}$. Let us consider a map f : $(X, \tau_1) \rightarrow (Y, \tau_2)$ defined by f (p) = r, f (q) = p and f (r) = q. Then we can verify that f is a rgw-homeomorphism. But, f is not a homeomorphism.

Theorem 3.4. Every rgw-homeomorphism is a g-homeomorphism.

Proof. Since every rgw-continuous map is g-continuous map. So easily can proof this.

Theorem 3.5. Every rgw∗-homeomorphism is a rgw-homeomorphism but not conversely. Proof. As we know that if f : $(X, \tau_1) \rightarrow (Y, \tau_2)$ is rgw-irresolute, then it is rgw-continuous and the fact that every rgw∗-open map is rgw-open.

Theorem 3.6. Let the function f : $(X, \tau_1) \rightarrow (Y, \tau_2)$ be a one-one onto rgw-continuous map. Then the following statements are equivalent:

(1) The function f is a rgw-open map.

(2) The function f is a rgw-homeomorphism.

(3) The function f is a rgw-closed map.

Theorem 3.7. For any topological space (X, τ), $h(X, \tau) \subseteq rgw-h(X, \tau)$.

Proof. Let us consider $f \in h(X, \tau)$, then by the definition of homeomorphism f and f^{-1} are continuous. Since every continuous function is rgw-continuous so f and f^{-1} is rgw-continuous map. Now by the definition of rgw-homeomorphism we can say that $f \in rgw-h(X, \tau)$.

Theorem 3.8. Let the function f : $(X, \tau_1) \rightarrow (Y, \tau_2)$ and g : $(Y, \tau_2) \rightarrow (Z, \tau_3)$ are rgw-homeomorphism, then composition of these two function as gof : $(X, \tau_1) \rightarrow (Z, \tau_3)$ is also rgw-homeomorphism.

Proof. Let G be rgw-open in Z. Now $(gof)^{-1}(G) = f^{-1}(g^{-1}(G)) = f^{-1}(G)$, where $V = g^{-1}(G)$. By given, if G us rgw-open in Z and g is rgw-homeomorphism, then $g^{-1}(G) = H$ is rgw-open in Y . Again, as given $g-1(G)$ is open in Y and f is rgw-homeomorphism, then $f^{-1}(g^{-1}(G))$ id rgw-open in X . Therefore, gof is continuous. Similarly we can show that $(gof)^{-1}$ is also continuous. Hence, by the definition gof is rgw-homeomorphism. By the above result we can proof that

Theorem 3.9. The rgw-homeomorphism is an equivalence relation in the family of all topological spaces.

Theorem 3.10. Let (X, τ) be a topological space, then the collection $rgw-h(X, \tau)$ forms a group under the composition of functions.

Proof. Let a binary operation o' : $rgw-h(X, \tau) \times rgw-h(X, \tau) \rightarrow rgw-h(X, \tau)$ defined by f o' g = gof for all $f, g \in rgw-h(X, \tau)$, where gof : $X \rightarrow X$ is composite maps of f and g such that $(gof)(x) = g(f(x))$ for all x in X .By the theorem 3.8, gof $\in rgw-h(X, \tau)$. The following properties hold by the collection $rgw-h(X, \tau)$.

(1) Associativity: Since the composition of maps is associative, so easily prove

(f o' g)o' h = f o'(g o' h) for all $f, g, h \in rgw-h(X, \tau)$

(2) Existence of identity: Since the identity map $i_X : X \rightarrow X$ is also rgw-homeomorphism, then easily we can say that for all element $f \in rgw-h(X, \tau)$, there exists an element i_X such that

$$f \text{ o' } i_X = i_X \text{ o' } f = i_X$$

(3) Existence of inverse: We know that the composition of maps is associative and the identity map $i_X : (X, \tau) \rightarrow (X, \tau)$

belonging to rgw-h(X, τ) servers as the identity element. If f ∈ rgw-h(X, τ), then f^{-1} ∈ rgw-h(X, τ) such that $f o f^{-1} = f^{-1} o f = i$ and so inverse exists for each element of rgw-h(X, τ).

Therefore, (rgw-h(X, τ), o) is a group under the operation of composition of maps.

Theorem 3.11. The homeomorphism group h(X, τ) is a subgroup of the group rgw-h(X, τ).

Proof. It is obvious that 1_X : (X, τ) → (X, τ) is a homeomorphism and so h(X, τ) ≠ φ.It follows from that h(X, τ)⊂ rgw-h(X, τ). Let a, b ∈ h(X, τ). Then we have that $o'(a, b^{-1}) = b^{-1} o a$ ∈ h(X, τ), here o' : rgw-h(X, τ) × rgw-h(X, τ) → rgw-h(X, τ) is the binary operation. Therefore, the group h(X, τ) is a subgroup of rgw-h(X, τ).

Next important result give the relation between homeomorphism of space with group isomorphism of their family of rgw-homeomorphism.

Theorem 3.12. If f : $(X, \tau_1) \to (Y, \tau_2)$ is homeomorphism, then there exists isomorphism between rgw− h(X, τ_1) and rgw−h(Y, τ_2) i.e. rgw−h(X, τ_1) ≅ rgw − h(Y, τ_2).

Proof. Using the map f , we define a map ψ_f : rgw − h(X, τ_1) → rgw − h(Y, ρ) by $\psi_f(h) = f o h o f^1$ for every h ∈ rgw − h(X, τ_1). Then ψ_f is bijection. Further, for all $h_1, h_2 \in \psi_f$, $\psi_f(h_1 o h_2) = f o(h_1 o h_2) o f^{-1} = (f o h_1 o f^{-1}) o (f o h_2 o f^{-1}) = \psi_f(h_1) o \psi_f(h_2)$. Therefore ψ_f is a homeomorphism and so it is an isomorphism induced by f .

Definition 3.13. A function f : $(X, \tau_1) \to (Y, \tau_2)$ is said to contra rgw-irresolute if f^{-1} is rgw-closed in (X, τ_1) for every rgw-open set H of (Y, ρ).

Lemma 3.14. Let two function f : $(X, \tau_1) \to (Y, \tau_2)$ and g : $(Y, \tau_2) \to (Z, \tau_3)$ defined on topological spaces (X, τ_1) and (Y, τ_2) respectively, then

(1) If functions f and g are contra rgw-irresolute, then the composition gof is also rgw-irresolute.

(2) If function f is rgw-irresolute (resp. contra rgw-irresolute) and g are contra rgw-irresolute (resp. rgw-irresolute), then the composition function gof is contra rgw-irresolute.

Definition 3.15. For a topological space (X, τ), we define the collection of functions contra-rgw-h(X, τ) as follows.

con − rgw-h(X, τ) = {f : f (X, τ) → (X, τ) is a contra rgw-irresolute bijection and f^{-1} is rgw-irresolute}

For a topological space (X, τ), we construct alternative groups, say rgw-h(X, τ) ∪ con −rgw-h(X, τ).

Theorem 3.16. If (X, τ) be a topological space, then union of two collections, rgw-h(X, τ)∪ con−rgw-h(X, τ), forms a group under the composition of functions.

Proof. Let us BX = rgw-h(X, τ)∪con−rgw-h(X, τ). A binary operation $w_X : B_X \times B_X \to B_X$ is well defined by $w_X(a, b)$ = boa, where boa : X → X is the composite function of the functions a and b. Indeed, let (a, b) ∈ BX ; if a ∈ rgw-h(X, τ) and b ∈ con−rgw-h(X, τ), then boa : (X, τ)→ (X, τ) a contra rgw-irresolute bijection and $(boa)^{-1}$ is also contra rgw-irresolute and so $w_X(a, b)$ = boa ∈ rgw-h(X, τ) ⊂ B_X , if a ∈ rgw-h(X, τ) and b ∈ rgw-h(X, τ) then boa : (X, τ) → (X, τ) is a rgw-irresolute bijection and so $w_X(a, b)$ = boa ∈ rgw-h(X, τ) ⊆ BX , if a ∈ con − rgw-h(X, τ) and b ∈ con − rgw-h(X, τ), then boa : (X, τ) → (X, τ) is a rgw-irresolute bijection and $(boa)^{-1}$ is also rgw-irresolute and so $w_X(a, b)$ = boa ∈ rgw-h(X, τ) ⊂ B_X is a ∈ con−rgw-h(X, τ) and b ∈ rgw-h(X, τ) then boa : (X, τ) → (X, τ) is a contra reg-irresolute bijection and $(boa)^{-1}$ is also rgw-irresolute and and so $w_X(a, b)$ = boa ∈ con − rgw-h(X, τ) ⊆ B_X . By the similar arguments, it is claimed that the binary operation $w_X : B_X \times B_X \to B_X$ satisfies the axiom of group; for the identity element e of B_X , e = 1_X : (X, τ) → (X, τ). Thus the pair (B_X , w_X) forms a group under the composition of functions, i.e., rgw-h(X, τ) ∪ con − rgw-h(X, τ) is a group.

Theorem 3.17. The homeomorphism group h(X, τ) is a subgroup of rgw-h(X, τ) ∪ con −rgw-h(X, τ).

Proof. By Theorem 3.11, it can be show that h(X, τ) is a subgroup of rgw-h(X, τ) ∪ con −rgw-h(X, τ).

Theorem 3.18. Let f : $(X, \tau_1) \to (Y, \tau_2)$ is rgw-continuous and collection of functions f is defined as Gg (f) = {(x, y) ∈ X × Y : y = f (x)}, where X × Y is product topological space and Gg (f) is called rgw-graph f . Then the following properties are satisfied:

(1) Gg (f), as a subspace of X × Y , rgw-homeomorphism to X .

(2) If Y is rgw-Housdrofff space, then Gg (f) is rgw-closed in X × Y .

Proof.

(1) Consider the function g : X → G_g (f) is defined by g(x) = (x, f (x)) for each x ∈ X is rgw-continuous and g^1 is also rgw-continuous. It is obvious that g is an injective function. Let P and Q are an arbitrary neighbourhood x ∈ X and (x, f (x)) in G_g(f) respectively. So, there exists two rgw-open sets U and V in X and Y respectively containing x and f (x) foe which (U × V) ∩ G_g (f) ⊂ E and U ⊂ P and f (U) ⊂ V . Let N = (U × V) ∩ G_g (f), then (x, f (x)) ∈ N and x ∈ g^{-1} (N) ⊂ U ⊂ P . This shows that g^{-1} is rgw-continuous. Therefore, g(U) ⊂ (U × V) ∩ G_g (f) ⊂ Q. Hence g is rgw-continuous which means that g is a rgw-homeomorphism.

(2) Let (x, y) ∉ Gg (f). Then y_1 = f (x) = y. By hypothesis, there exist disjoint rgw-open sets V_1 and V in Y such that $y_1 \in V_1$, y ∈ V . Since f is rgw-continuous, there exists an open set U in X containing x such that f (U) ⊂ V_1. Then g(U) ⊂ U × V_1. It follows from this and the fact that V_1 ∩ V = φ that (U × V) ∩ G_g (f) = φ.

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